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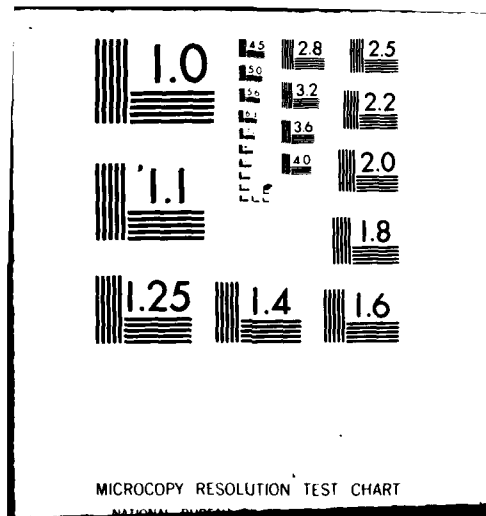
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ON TESTS FOR EQUICORRELATION COEFFICIENT AND THE
GENERALIZED VARIANCE OF A STANDARD SYMMETRIC
MULTIVARIATE NORMAL DISTRIBUTION

BY

ASHIS SEN GUPTA

TECHNICAL REPORT NO. 55

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OFFICE OF NAVAL RESEARCH

THEODORE W. ANDERSON, PROJECT DIRECTOR

DEPARTMENT OF STATISTICS
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On tests for equicorrelation coefficient and the
generalized variance of a standard symmetric
multivariate normal distribution

Ashis Sen Gupta
Stanford University

1. Introduction and Summary. A random vector follows a symmetric multivariate normal (SMN) distribution (Rao, 1973) if the components have equal means, equal variances and equal covariances -- the correlation coefficient, ρ , between any two components is the same and is termed the intraclass, equi-, uniform (Geisser, 1963) or familial (Fisher, 1925) correlation. Since they arise naturally in psychology, education, medicine, genetics etc., such models have received considerable attention, e.g. in the same year and in the same journal by Geisser & Desu (1968), Gleser (1968) and Han (1968) and very recently by Mak & Ng (1981). Again in bio-sciences (Press, 1981) many organisms e.g. starfish, octopus etc. exhibit a natural symmetry and consequently give rise to the above distribution. The importance of such models in many tests of multivariate analysis, e.g. in MANOVA, Profile Analysis, Growth curve analysis etc. has been established by Huynh and Feldt (1970). Wilks (1946) and Votaw (1948) have considered likelihood ratio tests for these models. Tests for ρ have been proposed by Srivastava (1965) using Roy's union intersection principle, by Aitken, Reinfurt and Nelson (1968) and by Mak & Ng using the likelihood ratio (LR) principle, by Rao (1973) using the canonical form of the SMN distribution and by Cokhale & SenGupta (1982) using the approach of locally most powerful (LMP) tests. These models have been investigated using the theory of products of problems by Arnold (1973) and in the general framework of representation theory and invariance by Andersson (1976). Recently, they have been generalized, e.g. by Szatrowski (1979) to incorporate block symmetries.

A random vector X will be said to follow a standard symmetric multivariate normal (SSMN) distribution if it follows a SMN distribution and

additionally the components have zero means and unit variances. Though the literature on the SMN distribution is quite extensive, yet little is known about SSMN distributions. In particular, no tests for ρ is known for SSMN distributions. However, such distributions can occur naturally in various ways (Sampson, 1978). One such situation is 'when there are many observations individually taken at different times on each scalar multivariate normal random variable, and then several vector observations are taken for all the variables together'. Another situation is 'when there are many missing observations for individual entries in the vector sample' and complete observation vectors are treated as coming from the distribution with 'known' means and variances found by using vectors with missing entries. Further, in many practical problems it is necessary to standardize the variables. Even the sample means and variances are usually employed for such standardizations and then the resulting variables behave asymptotically, by Slutsky's theorem, as standardized random variables. Such standardizations are always made and play important roles in the techniques for reduction of dimensionality, e.g. in canonical variables (Anderson, 1958) and generalized canonical variables analysis (Sen Gupta, 1981b,c). Finally, if the means and variances are known and hence can be considered as zeros and ones respectively, the discussions in the previous paragraph translates to the case of SSMN distributions. The present case is also quite interesting from several theoretical considerations. Firstly, it provides a practical example of Efron's curved exponential family (Efron & Hinkley, 1978) and illustrates some associated difficulties and techniques to overcome them. Secondly, observe that this situation arises when the components of X are exchangeable, i.e. the distribution of X is independent of permutation of its components. Also, the SSMN distribution constitutes an example of the mean-zero invariant model of Andersson.

Thirdly, it indicates (through ρ) the construction of simple, intuitive estimators of the correlation coefficient for such models in contrast to those obtained by the iterative method of scoring of Fisher or the sequential Robbins-Monro procedure suggested by Sampson (1976). Finally, it is demonstrated that unlike the LR tests, small sample optimal test for the correlation coefficient may be conveniently derived in such models. Though, as noted by Anderson (1963) 'the theory in the case of correlation matrices is much more complicated than for covariance matrices and no general result could be given in a simple form', we observe that for the above important and special structure of the correlation matrix, interesting and elegant results can be derived.

Several estimators and tests for ρ are considered. Like the bivariate case, the maximum likelihood estimator (MLE) is obtained as a root of a cubic equation and is shown to lie in the interval restricted by ρ . But, unlike the incomplete bivariate case (Dahiya & Korwar, 1980) it may be necessary to evaluate all real roots of this equation which makes the estimation cumbersome and the exact distribution of the estimator nearly intractable. This also renders the LR test statistic computationally quite inconvenient. Further, it is shown that the LR tests for one-sided alternatives are vacuous with positive probability. This serious drawback, coupled with the lack of knowledge regarding its small sample properties and computational inconvenience motivates the development of alternative tests. A test based on the best natural unbiased estimator (BNUE) of ρ is proposed. Also, a LMP test for ρ is derived and is seen to coincide with that based on the BNUE of ρ . This test is proved to be unbiased. The exact distribution of this test statistic, historically, happens to be a problem attempted by

Pearson et al (1932,p.341). The exact and asymptotic distributions are derived here. Percentage points are available. The case of constrained parameter space is considered.

The generalized variance, the determinant of the dispersion matrix, was proposed by Wilks (1932) as a scalar measure of multidimensional scatter. It has enjoyed applications in nearly all branches of applied statistics. But, tests and associated exact distributions (SenGupta, 1981a) become quite complicated in the general case. However, for the SSMN distribution some simplifications are available. The LR test, which again is computationally inconvenient, is derived. Alternatively, an elegant and simple test based on a characterization through the characteristic roots of the correlation matrix is proposed by introducing the concept of conditional characteristic roots.

2. Tests for equi-correlation coefficient ρ .

Let $\underline{Y}_1, \dots, \underline{Y}_m$ be an independent sample from $N_k(\underline{0}, \Sigma_\rho)$ where, letting I and E be the identity matrix and the matrix with all elements equal to unity respectively,

$$\Sigma_\rho = (1-\rho)I + \rho E; \Sigma_\rho^{-1} = (1-\rho)^{-1}I - \rho(1-\rho)^{-1}\{1+(k-1)\rho\}^{-1}E = (c_{ij}),$$

$$c_{ii} = \{1+(k-2)\rho\}/(1-\rho)\{1+(k-1)\rho\}, \quad c_{ij} = -\rho/(1-\rho)\{1+(k-1)\rho\}, \quad i \neq j.$$

Hence the density function for non-singular Σ_ρ can be written as

$$f(\underline{Y}; \rho) = \frac{1}{(2\pi)^{k/2} |\Sigma_\rho|^{1/2}} \exp \left[-\frac{1}{2} \left\{ \frac{(\sum y_i^2)}{(1-\rho)} + \frac{(\sum y_i)^2 (-\rho)}{(1+(k-1)\rho)(1-\rho)} \right\} \right] \dots \quad (2.1)$$

$$-\infty < y_i < \infty, \quad i = 1, \dots, k, \quad -1/(k-1) < \rho < 1.$$

The above representation is particularly useful because it shows that

- (i) the density function constitutes a member of Efron's curved exponential family
- (ii) there does not exist any one-dimensional sufficient statistic for ρ

(iii) $\sum (y_i - \bar{y})^2$ and $\bar{y} = \sum y_i/k$ are independent, and

(iv) the part of the exponent within the second bracket is monotonically decreasing in ρ , with positive probability.

We want to test $H_0: \rho = \rho_0$ against $H_1: \rho < (>) \rho_0$ or against $H_2: \rho \neq \rho_0$.

2.1 Likelihood ratio test. For testing H_0 against H_2 the LRT is derived below. The LRT can be performed even if $m = 1$. The likelihood function can be written from (2.1) above easily and differentiating this with respect to ρ and equating the derivative to zero we have

$$g(\rho) = (k-1)k\rho(1-\rho)\{1+(k-1)\rho + \sum_{ji} y_{ij}^2\{1+(k-1)\rho\}^2 + \sum_{ji} (\sum y_{ij})^2\{1+(k-1)\rho\}^2\} = 0,$$

where $y'_j = (y_{1j}, \dots, y_{kj})$, $j = 1, \dots, m$. This is a cubic equation in ρ , two of whose roots may be complex. Now, $g\{-1/(k-1)\}$, $g(1)$ and $g(0)$ are all positive (with probability one). Thus, it is not clear from $g(\rho)$ that the ML estimator of ρ , say $\hat{\rho}$, with $-1/(k-1) < \hat{\rho} < 1$, will always exist. However, it can be shown directly that $\lim_{\rho \rightarrow -1/(k-1)^+} f(y; \rho) = 0$ for $\rho \rightarrow \{-1/(k-1)\}^+$ or $\rho \rightarrow 1^-$ which also conforms to a general result of Anderson (1970). Since a cubic equation must have at least one real root it follows then that the ML estimator must be in the interval restricted to ρ . Further, in cases of several admissible solutions (at most three), by principle of Maximum Likelihood, we choose as the MLE of ρ , that which corresponds to the largest value of the Likelihood function and call it $\hat{\rho}$. Thus we get the following.

Theorem 1. Let $\underline{y} \sim N_k(\underline{0}, \Sigma_\rho)$. Then if $\underline{y}_1, \dots, \underline{y}_m$ constitutes an independent random sample from the above population, the Likelihood Ratio Test for testing $H_0: \rho = \rho_0$ against the alternative $H_2: \rho \neq \rho_0$, is given by

$$\text{Reject } H_0 \text{ iff } \lambda = \{|\Sigma_{\hat{\rho}}|/|\Sigma_{\rho_0}|\}^{m/2} \exp[-\frac{1}{2} \{a\{f_1(\rho_0) - f_1(\hat{\rho})\} + b\{f_2(\rho_0) - f_2(\hat{\rho})\}\}]$$

$< K$, a constant

where $\hat{\rho}$ is the MLE of ρ , $a = \sum_j \sum_i y_{ij}^2$, $b = \sum_j (\sum_i y_{ij})^2$, $f_1(\rho) = 1/(1-\rho)$, $f_2(\rho) = -\rho/[1+(k-1)\rho](1-\rho)$ and K is a constant to be determined so that the level of the test meets the specified value.

Under H_0 , for large m , $-2 \ln \lambda \sim \chi_1^2$. It is clear that the ML estimation for ρ is cumbersome and the exact distributions of $\hat{\rho}$ and the LR statistics are nearly intractable.

For one-sided alternatives, the small-sample behaviors of the LR tests can be even worse. For example, defining a test to be vacuous if the test statistic is a constant, we have,

Theorem 2. The likelihood ratio tests for testing $H_{01}: \rho = \rho_+$ ($0 < \rho_+ < 1$) against $H_{11}: \rho < \rho_+$ and also for testing $H_{02}: \rho = \rho_-$ ($-1/(k-1) < \rho_- < 0$) against $H_{12}: \rho > \rho_-$ are vacuous with positive probability.

Proof: Let $m = 1$, since similar proof holds for $m > 1$. Now,

$|\Sigma_\rho| = (1+(k-1)\rho)(1-\rho)^{k-1}$, $d|\Sigma_\rho|/d\rho = -k(k-1)\rho(1-\rho)^{k-2}$. So, $|\Sigma_\rho|^{-1} \uparrow$ for $0 < \rho < 1$ and \downarrow for $-1/(k-1) < \rho < 0$. Next, consider the representation of $\underline{Y}'\Sigma_\rho^{-1}\underline{Y}$ as in the portion within the second bracket in (2.1). Then

$$\begin{aligned} d(\underline{Y}'\Sigma_\rho^{-1}\underline{Y})/d\rho &= [\{1+(k-1)\rho\}^2(\Sigma y_i^2) + \{-1-(k-1)\rho\}^2(\Sigma y_i)^2]/\{(1+(k-1)\rho)(1-\rho)\}^2 \\ &= [c_1(\rho)(\Sigma y_i^2) + c_2(\rho)(\Sigma y_i)^2]/\{(1+(k-1)\rho)(1-\rho)\}^2 \\ &= h(\rho, \Sigma y_i^2, \Sigma y_i), \text{ say.} \end{aligned}$$

Consider testing H_{01} against H_{11} . Observe that if $\underline{Y}'\Sigma_\rho^{-1}\underline{Y} \geq \rho$ for $0 < \rho < 1$, then $\text{Sup } f(\cdot)$ under H_{11} will be attained at $\rho = \rho_+$ which also yields $\text{Sup } f(\cdot)$ under H_{01} . Hence, to prove the theorem for these hypotheses, it suffices to show that, $h(\cdot) < 0$ with positive probability.

By the reduction to the canonical form (Rao, 1973) for SSMN distribution, there exists an orthogonal transformation, $\underline{y} \rightarrow \underline{z}$, such that, $\sum y_i^2 = \sum z_i^2$ and $z_1 = \sum y_i / \sqrt{k}$, where $z_i, i=1, \dots, k$ are all independent. It follows that $z_1 \sim N(0, 1+(k-1)\rho)$ and $z_j \sim N(0, 1-\rho), j = 2, \dots, k$. Hence

$$P[h(\rho, \sum y_i^2, \sum y_i) < 0] = P[\{kc_2(\rho) + c_1(\rho)\} \{1+(k-1)\rho\} \chi_1^2 + (1-\rho)c_1(\rho) \chi_{k-1}^2 < 0]$$

where χ_1^2 and χ_{k-1}^2 are independent χ^2 variables with 1 and $k-1$ degrees of freedom respectively. To show that the above probability is positive, it suffices to show that, $kc_2(\rho) + c_1(\rho) = -(k-1)(1-\rho)^2$ is negative under H_{11} , which is obviously true. This establishes the theorem for testing H_{01} against H_{11} . Since $h(\cdot)$ is an indefinite quadratic form in Z_i 's, and $c_1(\rho) > 0$ and $c_2(\rho) < 0$, suitable modifications of the above arguments also establish the theorem for testing H_{02} against H_{12} . Due to the above difficulties we consider below several alternative tests for ρ .

2.2 Test based on BNUE of ρ . From (2.1), where $m=1$, it follows that $(\sum y_i^2, \sum y_i)$ is a sufficient statistic for ρ . But this is not complete, since, $E(\sum y_i) = 0$. Note that, $E(y_i y_{i'}) = \rho, i \neq i', i, i' = 1, \dots, k$. Further, $(\sum y_i)^2 - \sum y_i^2 = \sum_{i \neq i'} y_i y_{i'}$, so that based on a sufficient statistic a 'natural' unbiased estimator for ρ is $\sum_{i \neq i'} y_i y_{i'} / k(k-1)$. For $m > 1$, considering natural estimators of the form, $\sum_j a_j (\sum_{i \neq i'} y_{ij} y_{i'j} / k(k-1))$ it follows that the best (minimum variance) natural unbiased estimator (BNUE) of ρ is

$$\tilde{\rho} = \sum_j \sum_{i \neq i'} y_{ij} y_{i'j} / mk(k-1).$$

Hence a test for ρ against one or two-sided alternatives can be based on $\tilde{\rho}$. But it is known that a test based on a good estimator need not be a good test [e.g. a test for correlation coefficient in the bivariate case (Kendall & Stuart (1967))]. However, for the present case, it is reassuring to note the following desirable result.

Theorem 3. For testing $H_0: \rho=0$ against $H_1: \rho < (>)0$, the test which rejects H_0 iff, $\tilde{\rho} < (> c')c$, where $c(c')$ is determined to give the desired level of significance, is unbiased.

Proof: For the canonical form of SSMN distribution discussed in Theorem 2, using m similar orthogonal transformations on y_{ij} 's, $i=1, \dots, k$, one for each j , $j=1, \dots, m$, $\underline{y}_j \rightarrow \underline{z}_j$, we have,

$$\tilde{\rho} = \frac{\sum_j \{(\sum_i y_{ij})^2 - (\sum_i y_{ij}^2)\}}{mk(k-1)} = \frac{\sum_j \{k z_{1j}^2 - \sum_i z_{ij}^2\}}{mk(k-1)} \quad (2.2)$$

Then, $\tilde{\rho}$ is distributed as $\{(1+(k-1)\rho)(k-1)\chi_m^2 - (1-\rho)\chi_{m(k-1)}^2\}/mk(k-1)$

where χ_m^2 and $\chi_{m(k-1)}^2$ are independent χ^2 variables with m and $m(k-1)$ d.f. respectively. Under H_0 , the distribution of $\tilde{\rho}$ is the same as above, with $\rho=0$.

Consider $H_1: \rho > 0$. To prove the theorem in this case, it suffices to show that

$$(k-1)\chi_m^2 - \chi_{m(k-1)}^2 < (1+(k-1)\rho)(k-1)\chi_m^2 - (1-\rho)\chi_{m(k-1)}^2$$

which is clearly true. The proof for $H_1: \rho < 0$ follows similarly.

Note that, $-\infty < \tilde{\rho} < \infty$ and it may be desirable to consider the modified truncated estimator $\tilde{\tilde{\rho}}$, where

$$\tilde{\tilde{\rho}} = \begin{cases} -1/(k-1), & \tilde{\rho} \leq -1/(k-1) \\ \tilde{\rho} & , \quad -1/(k-1) < \tilde{\rho} < 1 \\ 1 & , \quad \tilde{\rho} \geq 1 \end{cases}$$

2.3 Locally most powerful test. No small sample optimal test for ρ seems to be known. One such test is presented in the following theorem.

Theorem 4. The unbiased tests, defined in Theorem 3, based on the best natural unbiased estimator $\tilde{\rho}$ of ρ , are also locally most powerful tests.

Proof: Let $\dot{\ell}_{\rho}$ denote the first derivative of the log-likelihood function with respect to ρ . Then a LMP test for testing $H_0: \rho=0$ against

$H_1: \rho > (<) 0$ is given by,

$$\dot{\ell}_0 > (<) k, \quad (2.3)$$

where k is to be determined to provide the desired level of the test.

Now, (2.3) is equivalent to

$$-\{\Psi_1(0) + \Psi_2(0)\} > (<) k'$$

where $\Psi_1(\rho) = d|\Sigma_{\rho}|/d\rho$ and $\Psi_2(\rho) = d(\Sigma \underline{y}_j \Sigma_{\rho}^{-1} \underline{y}_j)/d\rho$. Use of the expressions for $\Psi_1(\rho)$ and $\Psi_2(\rho)$ as given in proof of Theorem 2 and some simplifications establish the theorem.

Due to the remark (i) following (2.1), from Kallenberg (1981) we conclude that the shortcoming of the LMP test, under suitable conditions, tends to zero at the rate $m^{-1} |\log \alpha_m|^{3/2}$ where $\alpha_m \in (0,1)$ is the level of significance.

2.4. Exact null and non-null distribution of $\tilde{\rho}$. The exact distribution of $\tilde{\rho}$ is that of the weighted difference of two independent χ^2 variables with different weights and arbitrary d.f.s. Now, historically, this problem was attempted by Pearson et al (1932, p. 341) and later solved only partly for the very special case of equal weights and equal d.f.s by Pachares (1952). It was also encountered by Anderson (1963, p. 139) who conjectured a possible approximation. The distribution is presented below in terms of Kummer's function. For percentage points see Gokhale and SenGupta (1982).

Let $U(a,b;z)$ give independent solutions to the confluent hypergeometric differential equation of Kummer:

$$z d^2 \omega / dz^2 + (b-z) d\omega / dz - a\omega = 0.$$

Then, in terms of an integral

$$U(a,b;z) = [\Gamma(a)]^{-1} \int_0^{\infty} \exp(-zv) v^{a-1} (1+v)^{b-a-1} dv, \quad a > 0, \quad z > 0$$

and in terms of the ${}_1F_1$ hypergeometric function,

$$U(a,b;z) = \frac{\pi}{\sin \pi b} \left\{ \frac{{}_1F_1(a,b;z)}{\Gamma(1+a-b)\Gamma(b)} - \frac{z^{1-b} {}_1F_1(1+a-b,2-b;z)}{\Gamma(a)\Gamma(2-b)} \right\}$$

Theorem 5. Let $V = \alpha_1 V_1 - \alpha_2 V_2$ where $\alpha_1, \alpha_2 > 0$ and V_1, V_2 are independent χ^2 variables with v_1 and v_2 d.f. respectively. Then, the probability density function of V is given by,

$$\begin{aligned} f(v) &= [C(v_1, v_2)/\Gamma(v_1/2)] v^{(v_1+v_2-2)/2} \exp(-v/2\alpha_1) \\ &\quad \cdot U[v_2/2, (v_1+v_2)/2; \{(\alpha_1+\alpha_2)/2\alpha_1\}v], \quad v \geq 0 \\ &= [C(v_1, v_2)/\Gamma(v_2/2)] (-v)^{(v_1+v_2-2)/2} \exp(v/2\alpha_2) \\ &\quad \cdot U[v_1/2, (v_1+v_2)/2; \{-(\alpha_1+\alpha_2)/2\alpha_1\}v], \quad v \leq 0 \end{aligned}$$

$$\text{where } C^{-1}(v_1, v_2) = 2^{\frac{(v_1+v_2)/2}{\alpha_1} \frac{v_1/2}{\alpha_1} \frac{v_2/2}{\alpha_2}}.$$

$$\text{Proof: } \alpha_1 f(v) = \int_{-\infty}^{\infty} g_1\{(v+\alpha_2 v_2)/\alpha_1\} g_2(v_2) dv_2$$

where g_i represents the probability density function of V_i , $i=1,2$.

For $v \geq 0$, noting that the limits of the above integral reduce to 0 and ∞ ,

simplifications yield the form of $f(v)$ as in the theorem. For $v \leq 0$,

in order to represent $f(v)$ in terms of $U(\cdot)$ consider the following. First

note that the exponent for $U(\cdot)$ must be negative. An initial transformation,

$\alpha_2 v_2 = -vy$ and then a further transformation $y-1 = z$ yields the claimed result.

Using the above theorem and (2.2) we have the following Corollary.

Corollary. The exact non-null distribution of $\tilde{\rho}$ is given by $f(v)$ of Theorem 5

with $v_1 = m$, $v_2 = m(k-1)$, $\alpha_1 = \{1+(k-1)\rho\}/mk$ and $\alpha_2 = (1-\rho)/mk(k-1)$ from

which the null distribution is obtained by substituting $\rho = 0$.

2.5 Asymptotic null and non-null distributions of $\tilde{\rho}$. From the representation of $\tilde{\rho}$ in Section 2.2 as a weighted difference of two independent χ^2 variables, it follows that

$$E(\tilde{\rho}^h) = \sum_{j=0}^h (-1)^{h-j} \binom{h}{j} \alpha_1^j \alpha_2^{h-j} \mu_{1,j}' \mu_{2,h-j}', \quad h=1,2,\dots$$

where α_1 and α_2 are given in Section 2.4, $\mu_{1,s}' = E(\chi_m^{2s})$ and $\mu_{2,s}' = E(\chi_{m(k-1)}^{2s})$, $s = 0,1,\dots,h$. Recalling that $\tilde{\rho} = \sum_{j=1}^m \{ \sum_{i \neq j} y_{ij} y_{i,j} / k(k-1) \} / m$, by Central limit theorem, we have

Theorem 6. $k(m/2)^{1/2}(\tilde{\rho}-\rho)$ is distributed asymptotically as a normal variable with mean 0 and variance $\{1+(k-1)\rho\}^2 + (1-\rho)^2/(k-1)$.

2.6 Tests for ρ under constrained parameter space. In many applied problems, in addition to having SMN or SSMN distribution, further information on ρ may be available. For large k , we need $\rho > 0$ in order that Σ_ρ be positive definite. Again, e.g., in the problem of psychological testing theory considered by Wilks (1946) or in testing (Box, 1950) the Model II assumptions for a balanced one-way analysis of variance, etc. it is natural to require $\rho > 0$. Gleser and Olkin (1969) have discussed these examples and derived likelihood ratio tests for ρ under the restriction $\rho > 0$ on the parameter space. We consider the same problem for SSMN distributions. As noted earlier, even with unconstrained parameter space, the likelihood ratio test suffers from several undesirable properties. We propose below an alternative test based on the characteristic roots of the sample correlation matrix, study its unbiasedness and provide the null and non-null asymptotic distributions of the test statistic. Through obvious modifications, the same procedure can be applied to the case of $\rho < 0$.

Let the smallest characteristic root of Σ_ρ be λ . Then for $\rho > 0$, $\lambda=1-\rho$ with multiplicity $k-1$. We may estimate λ by $1-\bar{\rho}$ where $\bar{\rho}$ is a

modified estimator of ρ , e.g. the modified maximum likelihood estimator or the modified BNUE, modified by the restriction $\rho > 0$. Alternatively consider estimating λ directly. Now $\bar{d} = \sum_{i=2}^k d_i / (k-1)$ is an intuitive estimator of λ , where $d_1 \geq \dots \geq d_k$ are the characteristic roots of the sample correlation matrix. Further, from Anderson (1963), $[(m-1)^{1/2}(\bar{d}-\lambda) / \{\lambda(k-(k-1)\lambda)\}] [k(k-1)/2]^{1/2}$ is asymptotically a standard normal variate.

Theorem 7. Let $\underline{Y}_1, \dots, \underline{Y}_m$ be a random sample from the SSMN distribution with $\rho > 0$. Then

- (a) Testing $H_0: \rho = \rho_0$ against $H_1: \rho > (<) \rho_0$ is equivalent to testing $H'_0: \lambda = \lambda_0 = 1 - \rho_0$ against $H'_1: \lambda < (>) \lambda_0$.
- (b) The test which rejects H_0 in favor of $H_1: \rho > (<) \rho_0$ if and only if $\bar{d} < (> d'_0)$ $d_0, d_0(d'_0)$ a suitable constant, is asymptotically unbiased for all ρ satisfying $0 < \rho_0 < \rho < (k-2)/2(k-1)$ $[(k-2)/2(k-1) < \rho < \rho_0 < 1]$.

Proof:(a) First note that for a $k \times k$ correlation matrix, $(k-1)$ roots are all equal if and only if it is of the form Σ_ρ . The if part is trivial. The only if part follows from Anderson (1963, Appendix A). The desired equivalence then follows trivially.

- (b) Consider $H_1: \rho > \rho_0 > 0$. We use the asymptotic distribution of \bar{d} given above. Then, for large m ,

$$P(\bar{d} < d_0 | H_0) = \alpha \Leftrightarrow d_0 = (-\tau_\alpha) \lambda_0 \{k - (k-1)\lambda_0\} [2/k(k-1)(m-1)]^{1/2} + \lambda_0$$

The proposed test will be asymptotically unbiased if, for large m ,

$$P(\bar{d} < d_0 | H_1) > \alpha, \text{ i.e. if, for all } \lambda_1 < \lambda_0,$$

$$(-\tau_\alpha) \lambda_0 \{k - (k-1)\lambda_0\} [2/k(k-1)(m-1)]^{1/2} + (\lambda_0 - \lambda_1)$$

$$> (-\tau_\alpha) \lambda_1 \{k - (k-1)\lambda_1\} [2/k(k-1)(m-1)]^{1/2}$$

which reduces to the condition $\lambda_0 \{k - (k-1)\lambda_0\} < \lambda_1 \{k - (k-1)\lambda_1\}$. Let

$g(\lambda) = \lambda \{k - (k-1)\lambda\}$. Then $g(\lambda) \downarrow \lambda$ if $\lambda > k/2(k-1)$ or $\rho < (k-2)/2(k-1)$, as claimed.

The proof for $H'_1: 0 < \rho < \rho_0$ follows as in above with obvious modifications.

3. Tests for the generalized variance $|\Sigma_\rho|$. As stated in Section 1 and also discussed by Eaton (1967) tests for generalized variance are often of practical importance and interest. We consider several tests for $|\Sigma_\rho|$.

3.1 Likelihood ratio test. First note that, $|\Sigma_\rho| < 1$ for $\rho \neq 0$. Next, consider the following

Lemma 1. If $|\Sigma_\rho| = \sigma_0^2 < 1$, then there are precisely two real distinct solutions for ρ .

Proof: Recall (from the proof of Theorem 2) that $|\Sigma_\rho| \uparrow$ for $-1/(k-1) < \rho < 0$ and \downarrow for $0 < \rho < 1$. Hence, due to strict monotonicity there are two real solutions, say $\rho_2 > 0 > \rho_1$ to $|\Sigma_\rho| = \sigma_0^2$.

Then (using MLE of ρ from Section 2.1) we have

Theorem 8. The likelihood ratio test for $H_0: |\Sigma_\rho| = \sigma_0^2$ against $H_1: |\Sigma_\rho| \neq \sigma_0^2$ is given by

$$\text{Reject } H_0 \text{ iff } \lambda_m = (|\hat{\Sigma}_\rho|/\sigma_0^2)^{m/2} \exp\{-\frac{1}{2} \sum_{j=1}^m \underline{y}_j' (\Sigma_{\rho^{**}}^{-1} - \Sigma_{\hat{\rho}}^{-1}) \underline{y}_j\} < C$$

where ρ^{**} is such that $\prod_{j=1}^m f(\underline{y}_j; \rho^{**}) = \max_{\rho_1, \rho_2} \prod_{j=1}^m f(\underline{y}_j; \rho)$, $\hat{\rho}$ is the MLE of ρ and C is a constant to be determined such that the test has the desired level.

Under H_0 , for large m , $-2 \ln \lambda_m \sim \chi_1^2$.

The LR test statistic is computationally cumbersome. For one-sided alternatives, it becomes even more repulsive. Alternatively, another test is considered below.

3.2 Tests based on the smallest characteristic root. Let ρ_1 and ρ_2 be as defined in Proof of Lemma 1. Now, the distinct characteristic roots (c.r.) for Σ_{ρ_i} are $1-\rho_i$ and $1+(k-1)\rho_i$, $i = 1, 2$. Thus the following definition is introduced, noting that for $(k-1)\rho_i \neq -\rho_j$, the c.r.'s are all different, $i \neq j$, $i, j=1, 2$.

Definition. $1-\rho_i, 1+(k-1)\rho_i, i=1,2$ are defined as the conditional characteristic roots (c.c.r) of Σ_ρ subject to $|\Sigma_\rho| = \sigma_0^2$, where $\rho_i, i=1,2$ are the two distinct real solutions to $|\Sigma_\rho| = \sigma_0^2$.

Lemma 2. The c.c.r.s. are all different for $0 < \sigma^2 < 1, k > 2$.

Proof: It suffices to check $C_i: (k-1)\rho_i \neq -\rho_j, i \neq j, i,j=1,2$.

Consider C_1 . Suppose, if possible, $(k-1)\rho_1 = -\rho_2$. Then, from Lemma 1, the defining relation between ρ_1 and ρ_2 , i.e. $|\Sigma_{\rho_1}| = |\Sigma_{\rho_2}|$ reduces to

$$\Psi(\rho_2) = [\{1+\rho_2/(k-1)\}/(1-\rho_2)]^{k-1} - [\{1+(k-1)\rho_2\}/(1-\rho_2)] = 0.$$

To establish C_1 it suffices to show that $\Psi(\rho)$ has no real solution in $(0,1)$. Now, letting

$$\Psi(\rho) = f_1^{k-1}(\rho) - f_2(\rho), f_1(\rho) = \{1+\rho/(k-1)\}/(1-\rho), f_2(\rho) = \{1+(k-1)\rho\}/(1-\rho)$$

we have

$$\Psi'(\rho) = [f_1^{k-2}(\rho)-1]k/(1-\rho)^2.$$

But, $\rho > 0 \iff f_1(\rho) > 1 \iff \{f_1^{k-2}(\rho) > 1, k > 2\} \iff \Psi'(\rho) > 0$,

so that Ψ is strictly monotonically increasing for $0 < \rho < 1$. The desired result then follows by noting that $\Psi(\rho)$ is continuous at $\rho = 0$ with $\Psi(0) = 0$.

A similar argument yields C_2 since $0 < f_1(\rho) < 1$ for $-1/(k-1) < \rho < 0$.

For $k=2, \rho_1 = -\rho_2$ so that, testing $|\Sigma_\rho|$ is equivalent to testing ρ^2 .

Thus, in the following sequel, it is assumed that $k > 2$. Then, the smallest c.c.r., s.c.c.r($\Sigma_\rho || \Sigma_\rho| = \sigma_0^2, \sigma_0^2$ known) is uniquely defined by virtue of Lemma 2 and it will be denoted by $\lambda(\rho, \sigma_0)$.

Lemma 3. If $|\Sigma_Y| = \sigma_1^2 < \sigma_0^2 = |\Sigma_\rho| < 1$, then $\lambda(\gamma, \sigma_1) < \lambda(\rho, \sigma_0)$.

Proof: First consider, $|\Sigma_\rho| = \sigma_0^2$ and let $\rho_1 < \rho_2$ be the two real solutions. $1+(k-1)\rho_1$ and $1-\rho_2$ are less than both $1-\rho_1$ and $1+(k-1)\rho_2$ since $\rho_1 < 0 < \rho_2$. Then the s.c.c.r is either $1+(k-1)\rho_1$ or $1-\rho_2$.

Next, observe that $\lambda(\rho, \sigma_0)$ and $\lambda(\gamma, \sigma_1)$ must be of the same functional form. Consider, e.g., the case, $\lambda(\rho, \sigma_0) = 1+(k-1)\rho_1$, i.e. $(k-1)\rho_1 < -\rho_2$. If possible, suppose $\lambda(\gamma, \sigma_1) = 1-\gamma_2$, i.e. $(k-1)\gamma_1 > -\gamma_2$. Here k is fixed. Since $\lambda(\delta, \sigma^2)$, $-1/(k-1) < \delta < 1$, is a continuous function of σ^2 , for $(k-1)\gamma_1 > -\gamma_2$ to hold, there must exist a η such that $(k-1)\eta_1 = -\eta_2$, which contradicts Lemma 2.

It remains to show that $1+(k-1)\rho_1 > 1+(k-1)\gamma_1$ and $1-\rho_2 > 1-\gamma_2$. This follows from the strict monotonicity of $|\Sigma_\delta|$.

Theorem 9. Testing $H_0: |\Sigma_\rho| = \sigma_0^2$ against $H_1: |\Sigma_\rho| = \sigma_1^2 > \sigma_0^2$ is equivalent to testing $H'_0: \lambda(\rho, \sigma_0) < \lambda(\rho, \sigma_1)$.

Proof: For a $k \times k$ correlation matrix, $(k-1)$ roots are all equal iff it is of the form Σ_ρ . The if part is trivial. The only if part follows from Anderson (1963, Appendix A) as also stated by Lawley (1963). The theorem then follows by noting Lemma 3.

The above theorem motivates tests for $|\Sigma_\rho|$ to be based on estimators of smallest characteristic root (s.c.r.). We propose the tests:

Reject $H_0: |\Sigma_\rho| = \sigma_0^2$ against $H_1: |\Sigma_\rho| < (>) \sigma_0^2$ iff $\hat{\lambda} < (>) \lambda_0$,
and against $H'_1: |\Sigma_\rho| \neq \sigma_0^2$ iff $\hat{\lambda} < \lambda'$ or $> \lambda''$

where $\hat{\lambda}$ is a suitable estimator of s.c.r. and λ_0, λ' and λ'' are chosen so as to give the desired level.

In the following sequel, Lemmas 1, 2 and 3 are exploited to provide some simple tests for $|\Sigma_\rho|$. Let $\hat{\lambda} = \text{Min.}(1+(k-1)\bar{\rho}, 1-\bar{\rho})$ where $\bar{\rho}$ is a suitable estimator of ρ , e.g. $\tilde{\rho}, \tilde{\rho}$ or $\hat{\rho}$. Consider $\bar{\rho} = \tilde{\rho}$ the truncated BNUE of ρ , due to the difficulties associated with MLE as seen in Section 2.

Then, for $0 < \lambda_0 < 1$,

$$P(\hat{\lambda} < \lambda_0) = P\{\tilde{\rho} < (\lambda_0 - 1)/(k-1) | \tilde{\rho} < 0\} P(\tilde{\rho} < 0) + P\{\tilde{\rho} > 1 - \lambda_0 | \tilde{\rho} > 0\} P(\tilde{\rho} > 0).$$

The null and non-null exact distributions can then be obtained from the distribution of $\tilde{\rho}$ given in Section 2.4, since from Lemma 2 the s.c.c.r. and hence ρ is uniquely specified under H_0 and H_1 . The unbiasedness of this test can be investigated in the same manner as in Section 2.3.

H_0 may motivate modified estimators of s.c.r. Suppose, e.g., H_0 specifies $\lambda(\rho, \sigma_0) = 1 + (k-1)\rho_1$. In this case, let us assume the population s.c.r. is of the form $1 + (k-1)\rho$, $\rho < 0$. Since, also $\lambda(\rho, \sigma_0)$ and $\lambda(\gamma, \sigma_1)$ must be of the same form as proved in Lemma 3, one may restrict to estimators of the form $1 + (k-1)\bar{\rho}$. One approach could be to use the modified BNUE or MLE of ρ , modified by the restriction $\rho < 0$. Another would be to consider the characteristic roots of the sample correlation matrix R . For example, $1 + (k-1)\rho_1$ being the smallest root also specifies that the s.c.r. in the population is of multiplicity 1. This enables us to use Anderson's (1963) result: $[(m-1)^{1/2}(\bar{d} - \lambda_2) / \{\lambda_2(k - q_2\lambda_2)\}] (kq_1q_2/2)^{1/2}$ is asymptotically distributed as a standard normal variable where the $k \times k$ population correlation matrix \bar{R} has q_1 roots equal to λ_1 and $q_2 = k - q_1$ roots equal to $\lambda_2 < \lambda_1$, $\bar{d} = \sum_{i=1}^k d_i/q_2$, $d_1 \geq \dots \geq d_k$ are characteristic roots of R based on a sample

of size m . Hence, for large samples, the test can be based on \bar{d} . For $\lambda(\rho, \sigma_0) = 1 + (k-1)\rho_1$ or $1 - \rho_2$, $q_2 = 1$ or $k-1$ respectively. Further, $\hat{\lambda}$ can be taken as \bar{d} and a test for $H_0: |\Sigma_\rho| = \sigma_0^2$ can be based on it as before. The asymptotic distribution of \bar{d} is completely specified under H_0 and H_1 whenever the corresponding s.c.c.r.s are defined.

4. Applications. The model in this paper assumed known means and variances. Wilks (1946) considered the same model with known means and unknown variances. Cox (1958) gives a practical example of known mean and known variance situation. Efron and Hinkley (1978) discuss the same model as in this paper, for the special case of $k = 2$ in some detail in terms of information and curvature. Goodman (1981) applies this model, with $k=2$, to the analysis of two-way contingency tables. The SSMN distribution is also often applied to problems in reliability. The methods discussed here are hoped to also indicate solutions to a variety of problems with patterned correlation matrices, e.g. multivariate intraclass correlation, autoregressive, moving average etc. models (Sampson (1978)). For detailed discussions and a new development using the notion of mean curvature, of the multiparameter LMP test see Sen Gupta and Vermeire (1981). This may be used to generalize the results of Section 2.3 to the multivariate intraclass correlation models.

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20. ABSTRACT

It is observed that the standard multivariate normal distribution with equicorrelation coefficient, say ρ , plays an important role in applied sciences. Tests for ρ are derived. The likelihood ratio test is computationally cumbersome, vacuous against one-sided alternatives with positive probability and the exact distribution of the test statistic is nearly intractable. Alternatively, a test based on the best 'natural' unbiased estimator of ρ is proposed. It turns out to be locally most powerful and globally unbiased against one-sided alternatives. The exact null and non-null distributions of the test statistic which are of historical interest are derived and the exact percentage points are available. Large sample approximations are also given. With constrained parameter space, a simple test for ρ based on the eigen-values of the sample correlation matrix is proposed. The null and non-null asymptotic distributions of the corresponding test statistic are given and the unbiasedness of the test is studied. Finally, we present the likelihood ratio test and a simple test based on the eigen values of the sample correlation matrix as tests for the generalized variance after establishing that they can be characterized through tests for ρ .

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